REPRESENTATION AS A POISSON TRANSFORM

BY

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It is the purpose of this paper to establish necessary and sufficient conditions that a function $f(x)$, defined for all real values of x, admit representation as a Poisson transform

$$(1) \qquad f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(t)}{1 + (x - t)^2}, \qquad -\infty < x < \infty,$$

where A is a nondecreasing function. If we ask in addition that A be bounded the problem has already been solved by Standish [2] in terms of an operator described in the paper [1]. The removal of the condition of boundedness seems to require techniques quite different from his. In §5 we establish a result concerning harmonic function which may be of independent interest.

1. The function $\hat{f}(z)$. In [1] it is established that a function $f(x)$ defined by the formula (1) necessarily admits analytic continuation into the strip $|y| < 1$ of the complex z-plane. If we denote the continuation of $f(x)$ by $f(z)$ then (1) continues to hold if x is replaced by z.

It is also shown in [1] that if $f(x)$ is defined by (1) then the function $\hat{f}(z)$ defined by

$$(1.1) \qquad \hat{f}(z) = -\frac{1}{\pi} \int_{0+}^{\infty} u^{-2}[f(z + u) - 2f(z) + f(z - u)]du$$

exists for all *real z* and also admits continuation into the strip $|y| < 1$. It is important for the sequel to know that the formula (1.1) persists for complex values of z in the strip $|y| < 1$.

For this purpose it is clearly enough that the integral

$$\int_{-\infty}^{\infty} \frac{f(z - u)}{1 + u^2} du$$

converges uniformly in compact subsets of the strip $|y| < 1$. Since (1) holds with x complex it suffices to verify this uniform convergence for the integral

$$I = \int_{-\infty}^{\infty} \frac{du}{1 + u^2} \int_{-\infty}^{\infty} \frac{dA(t)}{|1 + (z - u - t)^2|}.$$

We are assuming of course that A is nondecreasing.

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174

After an inversion of the order of integration and a change of variable I takes the form

$$I = \int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2] \, | \, 1 + (z - u)^2 \, |} \, .$$

If $z = x + iy$ lies in some compact subset S of the strip $|y| < 1$, then

$$|y|^2 \leqq 1 - \epsilon^2, \qquad\qquad\qquad |x| \leqq B,$$

where ϵ and B depend only on S. For such z

$$|\, 1 + (z - u)^2 \,| \geqq \epsilon^2 + (x - u)^2,$$

so that I is dominated by

$$\int_{-\infty}^{\infty} dA(t) \int_{-\infty}^{\infty} \frac{du}{[1 + (t - u)^2][\epsilon^2 + (x - u)^2]} \, .$$

Apart from a multiplicative factor this is the same as

$$\int_{-\infty}^{\infty} \frac{dA(t)}{(t - x)^2 + (1 + \epsilon)^2}$$

which converges uniformly in $|x| \leqq B$, since

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(t)}{t^2 + 1}$$

exists by hypothesis.

We have proved that *the integral in* (1.1) *converges uniformly in compact subsets of the strip* $|y| < 1$.

2. The representation theorem. Let $f(z)$ be analytic in the strip $|y| < 1$, and let $\hat{f}(z)$ be defined by (1.1). We define an operator $T_t f$ for $-1 < t < 1$ and $-\infty < x < \infty$ by

$$(2.1) \qquad (T_t f)(x) = \frac{1}{2} \left[f(x + it) + f(x - it) \right] + \frac{1}{2i} \int_{x-it}^{x+it} \hat{f}(u) du$$

whenever it has meaning.

The main result can now be stated.

THEOREM 2.1. *Let $f(x)$ be defined for all real values of x. In order that it admit the representation* (1) *with A a nondecreasing function it is necessary and sufficient that*

(i) *$f(x)$ admits analytic continuation into the strip $|y| < 1$ of the z-plane;*

(ii) *the integral in* (1.1) *converges uniformly in compact subsets of $|y| < 1$ to a function $\hat{f}(z)$;*

(iii) *$(T_t f)(x) \geqq 0$, $0 \leqq t < 1$, $-\infty < x < \infty$;*

(iv) $f(x+iy) = o(x^2)$, $|x| \to \infty$, *uniformly in the substrips* $|y| \leqq \delta$ *for each* $\delta < 1$.

3. **Proof of the necessity.** Suppose that $f(x)$ has the form (1). The necessity of condition (i) is established in [1] and that of (ii) in the preceding section of this paper.

We turn our attention to (iii). In view of the analyticity of f and $\hat{f}$ in the strip $|y| < 1$, $(T_t f)(x)$ can be written as

$$\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} f^{(2k)}(x) + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \hat{f}^{(2k)}(x).$$

By formula (4.1) of [1] this is in turn equal to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v dA(u)}{v^2 + (x-u)^2}, \qquad\qquad v = 1 - t$$

for $0 \leqq t < 1$. (iii) follows since A is nondecreasing.

As for condition (iv), we shall establish it under the condition $x \to -\infty$; the proof for $x \to +\infty$ is similar. We begin by recalling formula (1.3) of [1] which states (if we set $z_0 = 0$) that $f(z)$ can be written in the form

$$(3.1) \qquad\qquad f(z) = f(0) - 2z \int_{-\infty}^{\infty} F(t) \frac{1 - (t-z)t}{[1 + (z-t)^2]^2} dt,$$

where $F(t)$ is the bounded function

$$F(t) = \frac{1}{\pi} \int_{-\infty}^{t} \frac{dA(u)}{1 + u^2}.$$

Since $|y| \leqq \delta < 1$ it suffices by (3.1) to prove that

$$J(z) \equiv \int_{-\infty}^{\infty} F(t) \frac{1 - (t-z)t}{[1 + (z-t)^2]^2} dt$$

$$= o(x), \qquad\qquad x \to -\infty,$$

uniformly in $|y| \leqq \delta$. $J(z)$ can be written as $K(z) - zL(z)$, where

$$K(z) = \int_{-\infty}^{\infty} F(t) \frac{1 - (t-z)^2}{[1 + (z-t)^2]^2} dt$$

and

$$L(z) = \int_{-\infty}^{\infty} F(t) \frac{t - z}{[1 + (z-t)^2]^2} dt.$$

We shall prove that each of these functions is $o(1)$ as $x \to -\infty$, uniformly in $|y| \leqq \delta$, and thus will complete the proof of (iv).

In each of the integrals defining K and L let $z = x + iy$ and then make the

change of variable $u = t - x$. We get

$$K(z) = \int_{-\infty}^{\infty} F(u + x) \frac{1 - (u - iy)^2}{[1 + (u - iy)^2]^2} du$$

and

$$L(z) = \int_{-\infty}^{\infty} F(u + x) \frac{u - iy}{[1 + (u - iy)^2]^2} du.$$

Then, since $|y| \leq \delta < 1$,

$$|K(z)| \leq \int_{-\infty}^{\infty} |F(u + x)| \frac{1 + (|u| + \delta)^2}{(u^2 + 1 - \delta^2)^2} du$$

and

$$|L(z)| \leq \int_{-\infty}^{\infty} |F(u + x)| \frac{|u| + \delta}{(u^2 + 1 - \delta^2)^2} du.$$

Because F is bounded and $F(-\infty) = 0$ each of these is $o(1)$ as $x \to -\infty$.

4. **The sufficiency.** Suppose now that conditions (i)–(iv) of the theorem hold.

It is a consequence of them that the integral

(4.1)
$$\int_{-\infty}^{\infty} \frac{|f(u)|}{1 + u^2} du$$

is finite. For by (ii) if we let $z = 0$ the integral

$$\int_{1}^{\infty} \frac{f(u) - 2f(0) + f(-u)}{u^2} du$$

converges and by (iii) with $t = 0$ we have $f(u) \geq 0$, $-\infty < u < \infty$. This establishes that (4.1) is finite.

Consequently the function $u(x, t)$ defined by

(4.2)
$$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(-t)f(u)}{(x - u)^2 + t^2} du$$

exists and defines a function harmonic for $t < 0$, $-\infty < x < \infty$. Since f is continuous u takes on the boundary value $f(x)$ as $t \to 0-$, and since $f \geq 0$ the function u is non-negative for $t < 0$.

By (i) and (ii) both f and $\hat{f}$ are analytic in the strip $|y| < 1$. Consequently the function $(T_t f)(x)$ defined by (2.1) is harmonic in the region $-1 < t < 1$, $-\infty < x < \infty$. We shall show in §5 that $u(x, t)$ *is the harmonic continuation of* $(T_t f)(x)$ *into the half-plane* $t < 0$. Consequently the two functions together form a function $v(x, t)$ which is harmonic in $t < 1$ and which by condition

(iii) is non-negative for $t<1$. Hence by Herglotz' theorem for a half-plane $v(x, t)$ takes the form

$$(4.3) \qquad v(x, t) = K(1 - t) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1 - t)dA(u)}{(1 - t)^2 + (x - u)^2}$$

for $t<1$, where A is nondecreasing and K is a non-negative constant. If we let $t=0$ and use the fact that $v(x, 0) = u(x, 0-) = f(x)$ we obtain from (4.3) that

$$(4.4) \qquad f(x) = K + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dA(u)}{1 + (x - u)^2} \, .$$

Also by (4.3)

$$K = \lim_{t \to -\infty} \frac{v(o, t)}{1 - t} = \lim_{t \to -\infty} \frac{u(o, t)}{1 - t}$$

so by (4.2)

$$K = \lim_{t \to \infty} \int_{-\infty}^{\infty} \frac{f(u)}{u^2 + t^2} \, du.$$

Since (4.1) is finite we must have $K=0$. (4.4) reduces to (1) and the proof of sufficiency is complete.

It remains only to establish the preceding italicized remark concerning harmonic continuation.

5. **Proof of the harmonic continuation.** This requires that for $-1<t<0$ we have $u(x, t) = (T_t f)(x)$, or replacing t by $-t$ that for $0<t<1$ the following formula holds:

$$(5.0) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(u)}{(x - u)^2 + t^2} \, du = \frac{1}{2} \left[f(x + it) + f(x - it) \right] - \frac{1}{2i} \int_{x-it}^{x+it} \widehat{f}(u)du.$$

The verification of this constitutes the remainder of the paper. For future applications we state the result explicitly.

THEOREM 5.1. *Under hypotheses* (i), (ii) *and* (iv) *of Theorem 2.1 and the finiteness of* (4.1) *the formula* (5.0) *holds in the region* $0<t<1$, $-\infty<x<\infty$.

Let x be fixed and define $f(x+u) = g(u)$, $-\infty<u<\infty$. By (1.1) $\widehat{f}(x+u) = \widehat{g}(u)$. Therefore it is enough to establish (5.0) when $x=0$. We write this out explicitly:

$$(5.1) \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(u)}{u^2 + t^2} \, du = \frac{1}{2} \left[f(it) + f(-it) \right] - \frac{1}{2i} \int_{-it}^{it} \widehat{f}(u)du,$$

for $0<t<1$.

To prove this write the left-hand side of (5.1) as the limit as $R \to \infty$ of

$$I(R) = \frac{1}{2\pi i} \int_{-R}^{R} \left\{ \frac{1}{u - it} - \frac{1}{u + it} \right\} f(u)\, du$$

which can be written as

$$(5.2) \qquad I(R) = \frac{1}{2\pi i} \int_{-R-it}^{R-it} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i} \int_{R+it}^{-R+it} \frac{f(w - it)}{w}\, dw.$$

We rewrite the first term on the right-hand side as follows. Consider the rectangular contour with vertices at $-R$, $-R-it$, $R-it$, R and indented by a semi-circle of radius ρ below the origin. Since $f(w+it)/w$ is analytic in and on this contour we have

$$\frac{1}{2\pi i} \int_{-R-it}^{R-it} = \frac{1}{2\pi i} \int_{-R-it}^{-R} + \frac{1}{2\pi i} \int_{R}^{R-it} + \frac{1}{2\pi i} \int_{C},$$

where C denotes the part of the contour running from R to $-R$ and indented below the origin. Now let $\rho \to 0$. Then $\int_C$ becomes the principal value integral P.V. $\int_{-R}^{R}$ plus $2\pi i$ times half the residue of $f(w+it)/w$ at $w=0$. Hence

$$\frac{1}{2\pi i} \int_{-R-it}^{R-it} \frac{f(w + it)}{w}\, dw = \frac{1}{2\pi i} \int_{-R-it}^{-R} + \frac{1}{2\pi i} \int_{R}^{R-it} + \frac{1}{2\pi i} \text{ P.V.} \int_{-R}^{R} + \frac{f(it)}{2}.$$

There is a similar formula for the second term on the right-hand side of (5.2). Adding the two we get

$$I(R) = \frac{1}{2} \left[f(it) + f(-it) \right] + \frac{1}{2\pi i} \text{ P.V.} \int_{-R}^{R} \frac{f(w + it) - f(w - it)}{w}\, dw$$

$$+ \frac{1}{2\pi i} \int_{-R-it}^{-R} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i} \int_{-R}^{-R+it} \frac{f(w - it)}{w}\, dw$$

$$+ \frac{1}{2\pi i} \int_{R}^{R-it} \frac{f(w + it)}{w}\, dw + \frac{1}{2\pi i} \int_{R+it}^{R} \frac{f(w - it)}{w}\, dw.$$

According to hypothesis (iv) we can in the denominators of the last two integrals replace w by R and in the preceding two by $-R$, with an error of $o(1)$ as $R \to \infty$. Hence

$$I(R) = \frac{1}{2} \left[f(it) + f(-it) \right] + I_1 + I_2 + I_3 + o(1), \qquad R \to \infty,$$

where

$$I_1 = \frac{1}{2\pi i} \text{ P.V.} \int_{-R}^{R} \frac{f(w + it) - f(w - it)}{w}\, dw,$$

$$I_2 = -\frac{1}{2\pi i R} \left\{ \int_{-R-it}^{-R} f(w + it)\, dw + \int_{-R}^{-R+it} f(w - it)\, dw \right\}$$

and

$$I_3 = \frac{1}{2\pi i R} \left\{ \int_R^{R-it} f(w + it)dw + \int_{R+it}^R f(w - it)dw \right\},$$

respectively. Since $I(\infty)$ is the left-hand side of (5.1) what remains to be proved is that $L = I_1 + I_2 + I_3 + o(1)$, $R \to \infty$, where

$$L = -\frac{1}{2i} \int_{-it}^{it} \widehat{f}(u)du.$$

By formula (1.1) and an integration by parts we have

$$\widehat{f}(z) = \frac{1}{\pi R} \left[f(z + R) + f(z - R) \right] - \frac{1}{\pi} \int_{0+}^R \frac{1}{u} \left[f'(z + u) - f'(z - u) \right] du$$

$$+ o(1), \qquad\qquad\qquad\qquad R \to \infty,$$

uniformly in compact subsets of $|y| < 1$. Hence, integrating from $-it$ to it we have

$$L = -\frac{1}{2\pi i} \frac{1}{R} \int_{-it}^{it} f(z + R)dz - \frac{1}{2\pi i} \frac{1}{R} \int_{-it}^{it} f(z - R)dz$$

$$+ \frac{1}{2\pi i} \int_{0+}^R \frac{1}{u} \left[f(it + u) - f(it - u) - f(-it + u) + f(-it - u) \right] du$$

$$+ o(1), \qquad\qquad\qquad\qquad R \to \infty.$$

The third term on the right-hand side is the same as I_1. We shall identify the first with I_3, and the second with I_2. This will complete the proof.

The first term can be written

$$-\frac{1}{2\pi i} \frac{1}{R} \left\{ \int_0^{it} f(z + R)dz + \int_{-it}^0 f(z + R)dz \right\}$$

$$= -\frac{1}{2\pi i} \frac{1}{R} \left\{ \int_{R-it}^R f(w + it)dw + \int_R^{R+it} f(w - it)dw \right\}$$

which is clearly the same as I_3.

A similar argument shows the second term is the same as I_2.

References

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